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OPTIMAL PRICING, USE AND EXPLORATION OF UNCERTAIN NATURAL RESOURCE STOCKS

by

(10) Kenneth J. Arrow and Sheldon Chang

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OPTIMAL PRICING, USE AND EXPLORATION OF UNCERTAIN NATURAL RESOURCE STOCKS

by

Kenneth J. Arrow and Sheldon Chang[1]

Units of natural resources, to be called "mines", are assumed distributed over the unexplored territory according to a Poisson process in space. At any moment, total reserves, R, and unexplored land, X, are given. Society can determine the rate of consumption, c, and the rate of exploration (space per unit time), x. Reserves, R, are drawn down by consumption and increased by discoveries made during exploration; X is decreased by exploration. At any moment, the payoff to society is a concave function of c less a linear function of x. Payoffs in the future are discounted back to the present at a constant rate.

The optimal policy is characterized by a decreasing function, $R_B(X)$. When $R > R_B(X)$, there is no exploration. The shadow prices of reserves and of unexplored land both rise at the rate of interest, and consumption is such that its marginal utility equals the shadow price of reserves. When R decreases to $R_B(X)$, exploration occurs at an infinite rate; R is increased by random discoveries and X is decreased by the area explored, so that R and X are transformed into R' and X', respectively, with $R' > R_B(X')$. For large values of X, $R_B(X)$ is almost zero, and the shadow prices move in random cycles but show only a slight upward trend, thereby casting some light on the failure of mineral prices to rise at the market rate of interest.

The classic Hotelling [1931] model of exploitation of exhaustible resources assumes in its simplest form that the stock of the resource is known from the beginning. If there are no extraction costs, then the shadow prices associated with an optimal extraction policy rise at the rate of the market rate of interest. The only variable that has to be determined is the initial price, which then determines all future prices;

this depends upon the interaction of demand (or utility) considerations with the initial stock. Clearly in a competitive world, prices would have to rise at the rate of interest to keep resource-holders indifferent between extracting the resource now and later.

If there are extraction costs, then the market price is the sum of the marginal extraction cost and the rent on the scarce stock. The latter still obeys the rule of the preceding paragraph, so that the rent will rise at the rate of interest.

The experience of many minerals, most strikingly oil from 1950 to 1970, is that the theoretically-derived increase in prices is not observed. One might take a very long-run view of the world and say that the post-1970 increases are a sort of making up for lost time. Perhaps the Hotelling theory looks better when 1977 is compared with 1950. Still such an evasion gives the theory little value.

No doubt many explanations for the discrepancy are possible. But certainly one rather obvious one is that the stock of the resource is very far from known. Hence, new discoveries yielding upward revision of estimated reserves change the basis of calculation for the Hotelling rents. The predicted rise at the rate of interest is offset by repeated downward revision of the initial price in response to changing supply estimates.

But if the stocks are in fact uncertain, as evidence by repeated changes in estimates, then that uncertainty should be reflected in the initial planning. It is the aim of this paper to begin such an analysis.[2]

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[2] The analysis of extraction and consumption policy under conditions of uncertainty about the amount of reserves has been studied a number of times in the literature. For a survey, see Crabbe [1977], and for a recent example, see Loury [1976].

It is important, in my judgment, to distinguish between the process of learning about reserves and the consumption of them.[3] The first constitutes exploratory activity. It is clearly not the case that the only form of information about resource reserves comes from pumping for consumption.

The analysis in this paper is largely heuristic, and some points remain to be clarified, as will be seen.

1.   The Model of Uncertainty About Resources

Resources are assumed to be distributed randomly throughout the relevant area; they can discovered by costly exploration.

In particular, it will be assumed that there is no spatial correlation of resource distribution. In probability terms, the resource quantities in two overlapping areas are assumed to be independent random variables.[4] It turns that this assumption, together with the natural condition that resource pools are non-negative, severely restricts the range of possible distribution to be used. In particular, the resources

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[3] Pindyck [1978] has distinguished between exploration and extraction but not in the context of uncertainty. Gilbert [1978] has introduced exploration as a means of gaining information, as in the present paper, but his probability assumptions are different. Like Loury [1976], he assumes a given probability distribution for the total stock; extraction and exploration change the distribution only in the sense that the posterior distribution is conditioned on the statement that the total is at least as great as the amount extracted and explored for. This assumption is not grounded on the chance distribution of resources over the surface of the Earth.

The model of Deshmukh and Pliska [1978], developed independently of the present paper, has fundamentally the same probability assumptions (strictly speaking, theirs are more general). However, they assume that there is no limit on unexplored land and hence in the long run the resource is not exhaustible.

[4] Some support for this view can be found in the model of Menard and Sherman ([1975], p. 337).

must be located in discrete spots, randomly distributed over the area.
If the mines are equally likely to be found in all parts of the area, then
two natural assumptions are that the probability of finding a mine in an
area h is, for small h, proportional to h, while the probability of
finding two or more mines in a small area goes to zero more rapidly than
h. In symbols,

$$\lim_{h \to 0} \frac{1}{h} \text{ Prob (one mine in area } h) = \lambda,$$

$$\lim_{h \to 0} \frac{1}{h} \text{ Prob (two or more mines in area } h) = 0 ,$$

where λ is some positive constant.

As is well-known (see, e.g., Karlin and Taylor ([1975], pp. 22-26)),
these assumptions imply a Poisson distribution of mines. Specifically,
in an area A,

$$\text{Prob (m mines in area } A) = \frac{e^{-\lambda A}(\lambda A)^m}{m!} \quad (m = 0,\ldots) .$$

This process shall be applied in the following way. At any time
t, the $\underline{\text{rate of exploratory effort}}$, is the area explored per unit time. Let,

$x(t)$ = rate of exploratory effort at time t .

Then in the interval, $(t, t + dt)$, for small dt, the area explored
is $x(t)dt$. If we let $h = x(t)dt$, we are assuming,

(1) Prob (discovering one mine between t and $t + dt$) = $\lambda x(t)dt$.

The probability of discovering two or more mines in this small interval
is infinitesimal compared with (1) and can be disregarded.

The learning in this model is simple but not non-existent. It is true that exploration in any area gives no information about resources outside that area. However, the unexplored area is diminishing; the reserves are the sum of those known to be in the explored area plus a random variable representing reserves in the unexplored area, but the variance of the latter component is diminishing as the unexplored area diminishes. Let,

$$X(t) = \text{area unexplored at time } t.$$

By definition,

$$(2) \quad \dot{X}(t) = -x(t) \quad ,$$

where the dot denotes differentiation with respect to time, $X(0)$ is given and $X(t)$ is restrained to be non-negative; presumably, $X(t)$ approaches zero as t increases.

We make now one central simplifying assumption: all mines have the same resource quantities. This could be generalized by assuming that there is a distribution of mine sizes, for example, logarithmic normal. However, I believe that this generalization can be accommodated with little change in the following analysis. Without loss of generality, the amount of resource in each mine will be taken as 1.

Let,

$$R(t) = \text{known reserves at time } t \quad ,$$

$$N(t) = \text{number of mines discovered between } 0 \text{ and } t \quad ,$$

$$c(t) = \text{rate of consumption of the resource at time } t \quad .$$

The total consumption of the resource up to time t is, then,

$$\int_0^t c(t)dt \quad .$$

The known reserves at time t are the reserves at time 0 less consumption up to time t plus the number of mines discovered between 0 and t.

$$(3) \qquad R(t) = R(0) - \int_0^t c(t)dt + N(t) \quad .$$

The variable N(t) is a random variable; it jumps up by one unit in any small interval, (t, t + dt) with a probability given by (1). Therefore R(t) is also a random variable. From (3),

$$(4) \qquad \dot{R}(t) = -c(t) \quad \text{if} t \text{ is between jump points of } N(t) \quad .$$

R(t) is constrained to be non-negative everywhere.

2.   The Criterion Function

The maximand will be the usual integral of discounted utilities. Here, utility at any point of time depends on both the amount of the resource being discussed and on "goods in general." Since even the most important mineral requires a relatively small fraction of national income for production, it is reasonable to assume the absence of income effects, so that utility is linear in goods-in-general. It could, therefore, be ignored in analysis except that we will regard exploration as a costly activity. The inputs to exploration will be assumed to be goods in general. Let,

P = price of exploration in terms of goods-in-general .

$U(c)$ = utility of consumption of resource at any instant .

The units of measurement of U can be taken to be goods-in-general; hence, the utility generated at any instant is $U(c) - Px$, when the consumption is c and the rate of exploration is x.

The payoff to the economy is,

$$(5) \qquad \int_{0}^{\infty} e^{-\rho t}[U(c(t)) - Px(t)]dt \quad ,$$

where

ρ = rate of time discount of utilities .

However, since constraint (3) involves a random variable, the payoff (5) will be a random variable, since future consumption and exploration will depend upon the random discoveries made in the intervening period. The economy will therefore seek to maximize,

$$(6) \qquad E\{ \int_{0}^{\infty} e^{-\rho t}[U(c(t)) - Px(t)]dt \} \quad .$$

Criterion (6) is to be maximized with respect to the instruments, $c(t)$, $x(t)$ subject to the constraints that $X(t) \geq 0$, $R(t) \geq 0$ everywhere, with $X(t)$ and $R(t)$ being defined by (2) and (3) respectively.

3.   Analysis of the Optimal Policy

The methods are those of dynamic programming; see Bellman and Dreyfus [1962] for a heuristic exposition.

We consider the maximum value of (6) (under constraints) as a function of the initial values of the two state variables, R and X.

That is, we let,

(7) $V(R, X) = \max E\{\int_0^\infty e^{-\rho t}[U(c(t)) - Px(t)]dt\}$ when $R(0) = R$, $X(0) = X$.

Because time enters explicitly only through the exponential discount rate, the optimum starting from $t_o > 0$ and discounted back to t_o (rather than zero) would be the same function of $R(t_o)$, $X(t_o)$ as given by (7). Hence, $V(R,X)$ is the sum of the integral over the interval $(0,t_o)$ and the expected value of V at time t_o, discounted back to time 0.

(8) $V(R,X) = \int_0^{t_o} e^{-\rho t}[U(c(t)) - Px(t)] + e^{-\rho t_o}E\{V(R(t_o), X(t_o))\}$.

Note that $R(t_o)$ is a random variables, because its value depends on the number of discoveries in the interval $(0,t_o)$.

Equation (8) is valid if $c(t)$ and $x(t)$ have been chosen optimally. Alternatively, we may consider $c(t)$ and $x(t)$ to be variables over the interval $(0,t_o)$ and optimize over them, on the assumption that $V(R(t_o), X(t_o))$ is itself calculated on the assumption that an optimal policy is followed thereafter. Notice that $R(t_o)$ and $X(t_o)$ both depend on the choice of instruments over $(0,t_o)$, and hence the optimization must take account of the effects on both terms.

Now assume that $t_o = dt$, a number sufficiently small that c and x may be regarded as constant over $(0,dt)$. We can also disregard the discounting within this interval. Hence, the first term in (8) can be approximated by,

$[U(c) - Px]dt$.

The factor $e^{-\rho dt}$ is approximately $1 - \rho dt$. From (3), $R(dt)$ is a random variable. With c treated as constant,

$$R(dt) \simeq R(0) - cdt + N(dt) \quad ,$$

where the symbol, "$\simeq$" means "approximately equal." But in the small interval $(0, dt)$, the probability of more than one mine being discovered is negligible. From (1), $N(dt) = 1$ with probability approximately $\lambda x dt$, and $= 0$ with probability $1 - \lambda x dt$. Also, $X(dt) \simeq X(0) - xdt$. Therefore,

$$(9) \quad E\{V(R(dt), X(dt))\} \simeq (1 - \lambda x dt)V(R(0) - cdt, X(0) - xdt)$$
$$+ \lambda x dt V(R(0) + 1 - cdt, X(0) - xdt) \quad .$$

Assume in addition that V is a differentiable function of R and X. Let $V_R^o = \partial V / \partial R$ evaluated at $(R(0), X(0))$, $V_X^o = \partial V / \partial X$ evaluated at the same point, and V_R^1 and V_X^1 the two partial derivatives evaluated at $(R(0) + 1, X(0))$. Also, note that $R(0) = R, X(0) = X$.

$$V(R(0) - cdt, X(0) - xdt) \simeq V(R,X) - V_R^o cdt - V_X^o xdt \quad ,$$

$$V(R(0) + 1 - cdt, X(0) - xdt) \simeq V(R + 1, X) - V_R^1 cdt - V_X^1 xdt \quad .$$

Substitution into (9), some rearrangement, and discarding terms in $(dt)^2$ yield,

$$E\{V(R(dt), X(dt))\} \simeq V(R,X) - V_R^o cdt - V_X^o xdt + \lambda x dt \Delta V(R,X) \quad .$$

(The operator, Δ, means the difference between the function at $R + 1$ and at R, X being held constant.)

If we substitute this and the other results into (8), subtract $(1 - \rho dt) V(R, X)$ from both sides, divide through by dt, and let dt approach

zero, we find,

(10)
$$\rho V(R,X) = \max_{c,x} \{U(c) - Px - V_R^0 c - V_X^0 x + \lambda x \Delta V\} \quad.$$

Equation (10) will not hold throughout the (R, X)-plane. Note that no upper bound has been set to the exploration rate and that exploration costs are linear. Therefore, it is possible and, as will be seen shortly, true that there will be moments of infinitely rapid exploration. We may think of a whole area being explored instantaneously, with a corresponding cost which is finite in total but incurred at an infinite rate. In such a case, there is an instantaneous downward jump in the state variable X (unexplored land). Equation (10) will not hold in the region of infinitely rapid exploration.

Now maximize (10) with respect to c and x, the instrument values at time 0. For the consumption rate, c, ignore the corner possibility c=0; this is certainly legitimate if $U'(0) = +\infty$, i.e., if the resource is dispensable.

(11)
$$U'(c) = V_R^0 \quad.$$

The maximand in (10) is linear in x.─────────────────────────────

One cannot exclude the possibility of either a corner solution, x = 0, or of an infinitely rapid exploration; indeed, one of these two cases will hold almost everywhere. Clearly, from (10), the rate of exploration (x) would be infinite if,
$$P + V_X^0 < \lambda \ \Delta V,$$
and would be zero if,
$$P + V_X^0 > \lambda \ \Delta V.$$

Clearly, V_R^o and V_X^o are the shadow rents of the stock of reserves and the stock of unexplored land, respectively. We can drop the superscript O, since the analysis holds at any time t. Let,

$$p_R = \text{rent of reserves} = V_R^- \; ,$$

$$p_X = \text{rent of unexplored land} = V_X \; .$$

Then (11) can be rewritten,

(11') $$U'(c) = p_R \; ,$$

Let,

(12) $$A_\infty = \{ (R, X) \mid P + p_X < \lambda \, \Delta V \; \},$$

(13) $$A_o = \{ (R, X) \mid P + p_X > \lambda \, \Delta V \; \},$$

(14) $$B = \{ (R, X) \mid P + p_X = \lambda \, \Delta V \; \},$$

Then, from the earlier remarks,

(12') exploration is infinitely rapid for $(R, X) \; \varepsilon \; A_\infty$,

(13') $x = 0$ for $(R, X) \; \varepsilon \; A_o$.

When $(R. X) \; \varepsilon \; B$, it is so far possible that there will be exploration at a positive but finite rate; in any case, B is the boundary between the two regions A_o and A_∞.

Equation (11') is familiar; the shadow price of consumption must equal the shadow price of holding reserves. Relations (12'-13') are more novel. The left-hand sides in the definitions (12-14) represent the costs of exploration, direct costs and the use of unexplored land, which has a value due to its potential for resources. The right-hand sides represent the benefits, the gain in total utility (a finite increment, not a rate

of change) due to finding a mine multiplied by the probability of finding one per unit effort.

We first investigate the structure of the optimal policy and of the return function, V, in region A_o. Here, $x = 0$, c is given by (11'), and,

$$\rho V = U(c) - V_R c,$$

from (10). Differentiate with respect to R and with respect to X; since c maximizes the right-hand side, it can be regarded as constant in accordance with the "envelope theorem."

(15) $\rho V_R = -V_{RR} c, \quad \rho V_X = - V_{RX} c.$

Divide the first equation by the second.

$$V_R/V_X = V_{RR}/V_{RX}.$$

This partial differential equation holds throughout the domain A_o. As is well-known, it is equivalent to the statement that there exist functions $\overline{V}$ and R_E such that,

(16) $V(R, X) = \overline{V}[R + R_E(X)]$ for $(R, X) \in A_o$.

Without loss of generality, we may set

(17) $R_E(0) = 0.$

Note that $R_E(X)$ may be regarded as the <u>resource-equivalent</u> value of unexplored lands.

The importance of this result is that the function $\overline{V}$ is the return function for the optimal solution in the case where there are no unexplored lands, as can be seen by setting $X = 0$ in (16). This is a standard problem which is easily solved. Thus, in the case where $U(c) = \ln c$, it is easy to see that,

$$\overline{V}(R) = (\ln \rho - 1 + \ln R)/ \rho.$$

Then, in the domain A_o, the instrument variables, c and x, are determined by the relations, x = 0, and, from (11'),

$$U'(c) = V_R = \overline{V}' \ [R + R_E(X)].$$

The movements of the shadow prices in A_o can also be established easily. Note that,

$$V_{RR} = \overline{V}''[R + R_E(X)].$$

Then,

$$\frac{1}{p_R} \ \frac{dp_R}{dt} \ = \ \frac{1}{V_R} \ \frac{dV_R}{dt} = \frac{1}{V_R} \ V_{RR} \ \dot{R};$$

since there is no exploration, $\dot{R} = - c$, by (4). Then by the first equation in (15),

(18) $\qquad \dot{p}_R/p_R = \ \rho$ for $(R, X) \ \epsilon \ A_o.$

as in the usual Hotelling theory. From (16),

(19) $\qquad p_X = V_X = \overline{V}' \ R_E'(X);$

hence the ratio p_X/p_R depends only on X and is therefore constant along any given path. Hence, we must also have,

(20) $\qquad \dot{p}_X/p_X = \ \rho$ for $(R, X) \ \epsilon \ A_o.$

This regime continues, with R decreasing, until the trajectory hits B. Note that the behavior has been defined up to an as yet unknown function, R_E.

Now let us consider the determination of the return function, $V(R, X)$, in A_∞. Suppose the initial situation is $(R, X + dX)$. Exploration occurs at an infinite rate, so consumption can be disregarded. When the area dX is explored, the probability of a discovery is λdX to a first approximation. If a discovery is made, the point, $(R, X + dX)$ is transformed into $(R+1, X)$; if not, it is transformed into (R, X). The cost

incurred is P dX. Hence,

$$V(R, X + dX) \simeq (\lambda \ dX) \ V(R+1, X) + (1 - \lambda \ dX) \ V(R, X) - P \ dX.$$

Dividing through by dX and letting dX approach 0 yields,

(21) $V_X(R, X) = \lambda \ \Delta V \ (R, X) - P$ for $(R, X) \ \epsilon \ A_\infty$.

Let the boundary B be represented by the curve, $R = R_B(X)$, so that,

(22) $R > R_B(X)$ for $(R, X) \ \epsilon \ A_o$, $R < R_B(X)$ for $(R, X) \ \epsilon \ A_\infty$.

We wish to determine $R_B(X)$. Since the point $(R_B(X), X)$ is on the boundary of both A_o and A_∞, it follows by continuity that V_X is given both by (19) and (21).

(23) $\bar{V}'[R_B(X) + R_E(X)] \ R_E'(X) = \lambda \ \Delta V \ [R_B(X), X] - P.$

For a path starting from some (R, X), $R > R_B(X)$, $V(R, X) = \bar{V}[R + R_E(X)]$ is maximized when $R_E(X)$ is maximized. If R_E is given up to and including X, it follows that we wish to maximize $R_E'(X)$, since this act maximizes $R_E(X + dX)$. From (23), then,

(24) $R_E'(X) = \max\limits_{r} \ [\lambda \ \Delta V \ (r, x) - P]/\bar{V}'[r + R_E(X)],$

(25) $R_B(X)$ is the value of r which achieves the maximum in (24).

These equations, together with (21) and (16) completely describe the solution. Start with $V(R, 0) = \bar{V}(R)$, a known function. Having defined the functions, $V(R, X)$, $R_E(X)$ up to some point X for all R, we can calculate $R_B(X)$ and $R_E'(X)$ from (24) and (25). We can then define $V(R, X + dX)$ from (21) for $R < R_B(X)$, $R_E(X + dX) = R_E(X) + R_E'(X) \ dX$, and therefore $V(R, X+dX) = \bar{V}[R + R_E(X+dX)]$.

One last important point. It can be shown that R_B is a decreasing function. Suppose, to the contrary, that it were increasing in some interval. Start with a point $(R, X+dX) \ \epsilon \ B$, so that $R = R_B(X + dX)$. If exploration over dX produces no discovery, then we arrive at the point

(R, X); but $R = R_B(X + dX) > R_B(X)$, so that $(R, X) \in A_o$. If a discovery

is made, then we arrive at $(R+1, X)$, which certainly belongs to A_o. In

that case,

$$\Delta V(R, X) = \overline{V}[R + 1 + R_E(X)] - \overline{V}[R + R_E(X)].$$

In the maximization in (24), $R_E(X)$ is taken as given. Hence,

maximizing over r is equivalent to maximizing over $s = r + R_E(X)$. There-

fore, (24) becomes,

$$R_E'(X) = \max_s \ [\lambda \ \Delta\overline{V}(s) - P] \ /\overline{V}'(s).$$

and,

$$R_B(x) = s^* - R_E(X), \text{ where } s^* \text{ achieves the maximization above.}$$

But the maximand is now independent of X, so that s^* is independent of X.

Since $R_E(X)$ is certainly increasing (an increase in unexplored lands can

certainly never decrease total return), $R_B(X)$ must be decreasing, a contra-

diction to the original assumption that R_B is increasing.

That R_B is decreasing is a bit surprising at first. However, suppose

there is an infinite amount of unexplored territory, the case studied by

Deshmukh and Pliska [1978]. Under our special assumptions (linear costs of

exploration), it is easy to see that the optimal path starting with some

reserves R is to exhaust them along an optimal path (with prices rising

at the rate ρ). When $R = 0$, then exploration takes place at an infinite

rate until a discovery is made. There is no reason to explore earlier;

there is no information gained, since a discovery will always be made with

probability 1, and, in view of discounting, it pays to postpone exploration

until the last possible moment. When the discovery is made, exploration

ceases, and consumption again proceeds until the new reserves are exhausted.

In this case, the price rises at the rate ρ until exhaustion but drops abruptly upon discovery. Over a long period of time, price has no trend, though it varies periodically.

We may assume that in our model, with large X, the effect is approximately the same as infinite X. Clearly, with $U'(0) = +\infty$, one would not proceed to absolute exhaustion; with a finite X, there is always a finite probability that there are no discoveries to be made. But with large X, the probability of that event, $e^{-\lambda X}$, is very small, so that $R_B(X)$ would be expected to be very small. On the contrary, with small X, information about the remaining reserves becomes more important, so that exploration takes place at a higher level of reserves.

The price history will show fluctuations with little upward trend when X is large; presumably the upward trend is stronger as X approaches zero, but this requires a probabilistic analysis not yet performed.

4. Commentary.

The alternation of zero and infinite rates of exploration is an unfelicitious feature of the model. It is due to the assumption that exploration costs are flows and can be turned on and off without cost. If one added to the model the need for capital invested in exploration and production, there would be a tendency to smooth out exploration activities in order to make better use of the capital. But this would convert the problem into one with three state variables, which would be even more difficult.

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13. ABSTRACT

Units of natural resources, to be called "mines", are assumed distributed over the unexplored territory according to a Poisson process in space. At any moment, total reserves, R, and unexplored land, X, are given. Society can determine the rate of consumption, c, and the rate of exploration (space per unit time), x. Reserves, R, are drawn down by consumption and increased by discoveries made during exploration; X is decreased by exploration. At any moment, the payoff to society is a concave function of c less a linear function of x. Payoffs in the future are discounted back to the present at a constant rate.

The optimal policy is characterized by a decreasing function, $R_B(X)$. When $R > R_B(X)$, there is no exploration. The shadow prices of reserves and of unexplored land both rise at the rate of interest, and consumption is such that its marginal utility equals the shadow price of reserves. When R decreases to $R_B(X)$, exploration occurs at an infinite rate; R is increased by random discoveries and X is decreased by the area explored, so that R and X are transformed into R' and X', respectively, with $R' > R_B(X')$. For large values of X, $R_B(X)$ is almost zero, and the shadow prices move in random cycles but show only a slight upward trend, thereby casting some light on the failure of mineral prices to rise at the market rate of interest.